Relativistic theory of elastic deformable astronomical bodies: perturbation equations in rotating spherical coordinates and junction conditions

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In this paper, the dynamical equations and junction conditions at the interface between adjacent layers of different elastic properties for an elastic deformable astronomical body in the first post-Newtonian approximation of Einstein theory of gravity are discussed in both rotating Cartesian coordinates and rotating spherical coordinates. The unperturbed rotating body (the ground state) is described as uniformly rotating, stationary and axisymmetric configuration in an asymptotically flat space-time manifold. Deviations from the equilibrium configuration are described by means of a displacement field. In terms of the formalism of relativistic celestial mechanics developed by Damour, Soffel & Xu, and the framework established by Carter & Quintana the post Newtonian equations of the displacement field and the symmetric trace-free shear tensor are obtained. Corresponding post-Newtonian junction conditions at interfaces also the outer surface boundary conditions are presented. The PN junction condition is an extension of Wahr's one which is a Newtonian junction conditions without rotating.

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I. INTRODUCTION

The theory of elastic deformable bodies is of great importance for quantitative models for the free and forced motions of astronomical bodies (specially for the Earth). Historically the perturbed Newtonian Euler equation for an elastic deformable Earth was apparently first derived by Jeffreys and Vicente[1]. Some different forms of it have appeared later in the literature[2]. Such a local treatment of global geodynamics has been pursued especially by Wahr[3], Schastok[4], and Dehant & Defraigne[5] to describe the nutation of the Earth. Clearly all of these investigations just mentioned are fully within Newton's theory of gravity. The theory of elasticity is also of importance for the interpretation of data resulting from modern observational techniques such as Very-Long Baseline Interferometry (VLBI), Lunar and Satellite Laser Ranging (LLR & LSR), and all other kinds of observations where the positions of Earth-bound points should be described with high position. The normal modes (or quasi-normal modes) of the Earth or other astronomical bodies, such as white dwarfs or neutron stars is another field of important applications[6, 7, 8] as also the calculation of the time evolution of the (mass and current) multipole moments[9] of astronomical bodies.

Extending the Newtonian theory of motion of elastic deformable bodies to include relativistic effects presents a new and improved basis for further discussions of the such problems. Recently Xu, Wu and Soffel[10] developed such a general relativistic theory of elastic deformable astronomical bodies on the basis of the Damour-Soffel-Xu (DSX) formalism as the foundation of modern general relativistic celestial mechanics at the first post-Newtonian approximation of Einstein's theory of gravity [11, 12, 13, 14]. In [10] we discussed the post-Newtonian perturbations of a uniformly rotating, stationary, and axisymmetric elastic body in a rotating Cartesian coordinate system. The general perturbations of such a configuration are treated within the Carter-Quintana formalism[15, 16]. A central result was the post-Newtonian dynamical equation for the displacement field in Cartesian coordinates representing the post-Newtonian version of the well-known Jeffreys-Vicente equation[2]. However, for practical applications, the common way of dealing with such perturbations is to go to spherical [17] instead of Cartesian coordinates and then to expand the relevant quantities in terms of scalar, vector and tensor spherical harmonics or so-called generalized spherical harmonics.

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In this paper, we follow the route of our previous paper [10] to deduce the post-Newtonian perturbed local evolution equations and the perturbed Eulerian equation for the displacement field of an elastic astronomical body in rotating spherical coordinates. To this end the Eulerian variation of Einstein's energy-momentum conservation law is performed. The Newtonian version of our results (neglecting all $1/c^2$ terms) is in agreement with standard results from the literature (e.g., Ref.[17] after correction of a typographical mistake). Post-Newtonian junction conditions for the transition from one layer to another with different elastic properties that were not treated in [10] are also presented here. Such general relativistic junction conditions have been discussed in a different context[18 , 19]. However, the application to a post-Newtonian displacement field presented here is new. Corresponding Newtonian junction conditions can e.g., be found in Wahr[3].

Symbols and notations are taken from the DSX papers [11, 12, 13, 14]: the space-time signature is taken as $-+++$, spacetime indices go from 0 to 3 and are denoted by Greek indices, while spatial indices (1 to 3) are denoted by Latin indices. We use Einstein's summation convention for both types of indices, whatever the position of repeated indices. We shall often abbreviate the order symbol $O(c^{-n})$ simply by $O(n)$. Local coordinates $X^\alpha = (cT, X^a)$ are chosen in that reference system that moves with the body under consideration. The DSX scheme provides a description of the metric tensor $G_{\alpha\beta}$ in a local system with two metric potentials (W, W^a).

In Section II, the post-Newtonian perturbation equations of elastic astronomical bodies both in Cartesian [10] and spherical coordinates are presented. Special emphasis is given to the new derivation in spherical coordinates. In Section III, we discuss post-Newtonian junction and boundary conditions. In the last Section some conclusion can be found.

II. POST-NEWTONIAN PERTURBATION EQUATIONS OF ELASTIC DEFORMABLE ASTRONOMICAL BODIES

A. Perturbed equations in Rotating Cartesian coordinates

The formalism starts by considering some isolated relaxed body that rotates uniformly with angular velocity $\mathbf{\Omega}$ about its symmetry axis with respect to some global non-rotating coordinate system. Deviations from such an equilibrium configuration and the action of tidal forces in a gravitational N-body system are then described by means of perturbation theory. In that case one has to deal with at least three different coordinate systems: a global coordinate system $x^\mu = (ct, x^i)$ like the Barycentric Celestial Reference System (BCRS) that extends to infinity and where the dynamics of the N-body system can be formulated, a local 'non-rotating' system $X^\alpha = (cT, X^a)$ like the Geocentric Celestial Reference System (GCRS) (e.g., kinematically non rotating or at least slowly rotating with an angular velocity of post-Newtonian order with respect to the global system) and finally some local coordinate system $(\overline{X}^\alpha = (c\overline{T}, \overline{X}^a)$ with $\overline{T} = T)$ whose spatial coordinates co-rotate uniformly with the equilibrium configuration. The post-Newtonian perturbation equations of an elastic deformable astronomical body in rotating Cartesian coordinates have been presented recently[10]. Here, some of the results are summarized. The components of the metric tensor in rotating Cartesian coordinates read

$$\overline{G}_{00} = -\exp\left(-\frac{2\overline{W} + \overline{V}^2}{c^2}\right) + O(5)\,, \tag{2.1}$$

$$\overline{G}_{0a} = \frac{\overline{V}^a}{c} - \frac{4\overline{W}_a}{c^3} + O(5)\,, \tag{2.2}$$

$$\overline{G}_{ab} = \delta_{ab}\exp\left(\frac{2\overline{W}}{c^2}\right) + O(4)\,, \tag{2.3}$$

where

$$\overline{W} \equiv W + \frac{2WV^2}{c^2} - \frac{4W^bV^b}{c^2} + \frac{V^4}{4c^2} \tag{2.4}$$

and

$$\overline{W}_a \equiv R^{ab}\left(W_b - \frac{1}{2}V^bW\right)\,. \tag{2.5}$$

A bar on top of some quantity indicates that it refers to the rotating coordinate system $\overline{X}^\alpha$, otherwise it will refer to the 'non-rotating' local coordinate system X^α. W and W_a are the scalar and vector potential which describe the

metric in local 'non-rotating' coordinates. For the relaxed ground state of the body these potentials result entirely from the gravitational action of the body itself. For more details the reader is referred to Damour et al.[11]. V^b is the rotation velocity of the equilibrium configuration, R^{ab} is a time-dependent rotation matrix (defined by (2.15)–(2.16) of [10]), $\overline{V}^a = R^{ab}V_b$ and $\overline{\Omega}^b = R^{bc}\Omega^c$. Ω^c is the angular velocity with respect to 'non-rotating' coordinates ($V^a = \epsilon_{abc}\Omega^b X^c$, ϵ_{abc} is the completely antisymmetric Levi-Civita symbol of rank 3 with $\epsilon_{123} = +1$). The perturbed energy balance equation after a first time integration reads (see (4.30) and (3.31) of [10])

$$\begin{aligned} \delta\rho &= -\nabla\cdot(\rho\mathbf{s}) - \frac{1}{c^2}\left(\rho\overline{V}^a\dot{s}^a + (ps^a)_{,a} + 2\overline{W}_{,a}s^a\rho + 3\rho\delta\overline{W}\right) + O(4) \\ &= -\rho_{,a}s^a - \rho^*\Theta + O(4)\,, \end{aligned} \tag{2.6}$$

where ρ is the energy density, $\rho^* = \rho + p/c^2$ the chemical potential per unit volume, p is the isotropic pressure and s^a are the spatial components of the contravariant displacement field. The volume dilatation Θ is given by

$$\Theta = s^b{}_{,b} + \frac{1}{c^2}\left(\overline{V}^b\dot{s}^b + \epsilon_{bcd}\overline{\Omega}^c s^d\overline{V}^b + 3\overline{W}_{,c}s^c + 3\delta\overline{W}\right) + O(4)\,. \tag{2.7}$$

Here, $\delta\overline{W}$ is the Eulerian variation of $\overline{W}$ in rotating coordinates. The perturbed Eulerian equation takes the form (Eq.(4.32) of [10])

$$\begin{aligned} 0 &= \rho^*\left(1 + \frac{2\overline{W}_{\rm G}}{c^2}\right)\left(\ddot{s}_a + 2\epsilon_{abc}\overline{\Omega}^b\dot{s}^c\right) + \rho^*\Theta\overline{W}_{{\rm G},a} - \rho^* s^b_{,a}\overline{W}_{{\rm G},b} - \rho^*(\delta\overline{W}_{\rm G})_{,a} - \rho^* s^b\overline{W}_{{\rm G},ba} \\ &\quad -(\kappa\Theta\delta_{a\beta} + 2\mu s^\beta{}_a)_{;\beta} + \frac{1}{c^2}\left\{\rho^*\left[\overline{V}^a(\overline{V}^b\ddot{s}^b) + \overline{W}_{{\rm G},b}\dot{s}^b\overline{V}^a - 2\overline{V}^b\dot{s}^b\overline{W}_{{\rm G},a} + (\delta\overline{W})_{,\overline{T}}\overline{V}^a\right.\right. \\ &\quad \left.\left. +8\dot{s}^b\overline{W}_{[b,a]} - 4(\delta\overline{W}_a)_{,\overline{T}}\right] + \kappa\left(\Theta\overline{W}_{{\rm G},a} - \dot{\Theta}\overline{V}^a\right) + \left(\kappa(4\overline{W}\Theta + \overline{V}^b\overline{V}^c s^b_{,c})\right)_{,a}\right\} + O(4)\,, \end{aligned} \tag{2.8}$$

where the post-Newtonian geopotential $\overline{W}_{\rm G}$ is given by

$$\overline{W}_{\rm G} = \overline{W} + \frac{\overline{V}^2}{2}\,.$$

The "dot" stands for the derivative with respect to the time variable $\overline{T} = T$ and δ indicates the Eulerian variation. The elastic moduli κ and μ are the compression modulus and the shear modulus respectively; s_{ab} is the shear-stress tensor (a complete representation of s_{ab} is given in (4.26) of [10]). The term containing the shear-stress tensor reads (see also (4.33) of [10])

$$\begin{aligned} (2\mu s^\beta{}_a)_{;\beta} &= (2\mu s_{ba})_{,b} + \frac{1}{c^2}\left\{-(4\mu\overline{W}s_{ba} + 2\mu s_{ca}\overline{V}^c\overline{V}^b)_{,b} + (2\mu s_{ab})_{,T}\overline{V}^b\right. \\ &\quad \left. +2\mu(2\overline{W}_{,c}s_{ac} + \epsilon_{acb}\Omega^c s_{bd}\overline{V}^d)\right\}\,. \end{aligned} \tag{2.9}$$

Equation (2.8) is the post-Newtonian Euler equation for the displacement field $\mathbf{s}$, sometimes called the post-Newtonian Jeffreys-Vicente equation.

B. Perturbation equations in rotating spherical coordinates

1. *Unperturbed and perturbed projection tensor in rotating spherical coordinates*

The importance to formulate the perturbation equations in rotating spherical coordinates was already stressed in the Introduction. Note that both the Eulerian and the Lagrangian variation of a tensor do not necessarily preserve the tensor character, whereas the difference between them, the Lie derivative ($\mathcal{L}_\xi = \Delta - \delta$), does. This implies that one cannot simply transform the variational equations from Cartesian to spherical coordinates. For that reason all derivations of the perturbed energy equation and the post-Newtonian Jeffreys-Vicente equation have to be repeated in spherical coordinates. These rotating spherical coordinates (r,θ,ϕ) are defined by ($\overline{X} = r\sin\theta\cos\phi, \overline{Y} = r\sin\theta\sin\phi, \overline{Z} = r\cos\theta$).

The metric tensor in such rotating spherical coordinates ($\widetilde{X}^\mu = (c\widetilde{T}, r, \theta, \phi)$ with $\widetilde{T} = T$) takes the form

$$\widetilde{G}_{00} = -\exp\left(-\frac{2\widetilde{W} + \widetilde{V}^2}{c^2}\right) + O(6)\,, \tag{2.10}$$

$$\widetilde{G}_{0a} = \widetilde{G}_{a0} = D_{ac}\left(\frac{\widetilde{V}^c}{c} - \frac{4\widetilde{W}^c}{c^3}\right) + O(5)\,, \tag{2.11}$$

$$\widetilde{G}_{ab} = D_{ab}\exp\left(\frac{2\widetilde{W}}{c^2}\right) + O(4)\,, \tag{2.12}$$

where $\widetilde{V}^a = \left(\partial\widetilde{X}^a/\partial\overline{X}^b\right)\overline{V}^b$, $\widetilde{W}^a = \left(\partial\widetilde{X}^a/\partial\overline{X}^b\right)\overline{W}^b$, $\widetilde{W} = \overline{W}$, $\widetilde{V}^2 = \overline{V}^2$ and

$$D_{ab} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{2.13}$$

The corresponding inverse matrix D^{ab} satisfies $D_{ab}D^{bc} = \delta^c_a$. For the 3-dimension quantities $\widetilde{V}^a$ and $\widetilde{W}^a$ we define $\widetilde{V}_a = D_{ab}\widetilde{V}^b$, $\widetilde{W}_a = D_{ab}\widetilde{W}^b$.

In the following we will choose $\mathbf{\Omega} = \Omega\,\mathbf{e}_z$, with Ω being constant, so that

$$\widetilde{V}^r = \widetilde{V}_r = \widetilde{V}^\theta = \widetilde{V}_\theta = 0\,, \quad \widetilde{V}^\phi = \Omega\,, \quad \widetilde{V}_\phi = \Omega\, r^2\sin^2\theta \tag{2.14}$$

and $\widetilde{V}^2 = \Omega^2 r^2\sin^2\theta$. From these quantities the Christoffel symbols and the orthogonal projection tensor (projecting into the 3-space of an observer that moves with the corresponding material element) can be derived. In Carter and Quintana's formalism[15] a body is described by means of a bundle of time-like world-lines in a four-dimensional space-time manifold. A mapping into a three-dimensional manifold that is composed of the various material elements of the body identifies the various "particles" of the body. The projection of any tensor onto the local rest frame of matter is achieved with the orthogonal projection tensor

$$\gamma_{\mu\nu} = G_{\mu\nu} + \frac{U_\mu U_\nu}{c^2}\,. \tag{2.15}$$

This tensor acts as a positive metric tensor on the tangent subspace orthogonal to the flow vectors. The corresponding inverse metric tensor is given simply by its equivalent contravariant form, since it satisfies $\gamma^{\mu\nu}\gamma_{\nu\sigma} = \gamma^\mu_\sigma$. U^μ is the four-velocity of some material element as tangent vector to its world line and $\gamma_{\mu\nu}U^\nu = 0$.

In General Relativity, the unperturbed and perturbed states of a body are considered as two configurations in separate four-dimensional space-time manifolds. Usually one starts with canonical coordinates x^μ in both manifolds and $x^\mu \to x^\mu + \Delta x^\mu$ maps the coordinates of a material element in the reference state onto the coordinates of the same element in the perturbed state, where Δx^μ is the position coordinate displacement in 4-dimension space-time. The quantities $\xi^\mu \equiv \Delta x^\mu$ are called the four-dimensional displacement field. The Lagrangian variation is the variation of the field in terms of a coordinate system which is itself dragged along by the displacement Δx^μ; it is denoted by the symbol Δ. The Eulerian variation denoted by δ is the variation taken at a fixed point in 4-dimension space-time. The relation between these two kinds of variations is given by $\delta = \Delta - \mathcal{L}_\xi$, where $\mathcal{L}_\xi$ stands for the Lie derivative along the displacement field ξ^μ. The displacement field, for obvious reasons, will be defined in rotating coordinates. It is taken as $\xi^\mu = (0, \xi^a)$ (see [10]).

The Euler variation of $\widetilde{G}_{\mu\nu}$ (denoted by $h_{\mu\nu}$) results from (2.10):

$$h_{00} = \delta\widetilde{G}_{00} = \frac{2\delta\widetilde{W}}{c^2}\left(1 - \frac{2\widetilde{W} + \widetilde{V}^2}{c^2}\right) + O(6)\,, \tag{2.16}$$

$$h_{0a} = \delta\widetilde{G}_{0a} = -D_{ac}\left(\frac{4\delta\widetilde{W}_c}{c^3}\right) + O(5)\,, \tag{2.17}$$

$$h_{ab} = \delta\widetilde{G}_{ab} = D_{ab}\left(\frac{2\delta\widetilde{W}}{c^2}\right) + O(4)\,. \tag{2.18}$$

Other important quantities are the Eulerian variations of the 4-velocity and the projection tensor. They take the form

$$\delta U^0 = \frac{1}{c}\left(\delta\widetilde{W} + \widetilde{V}_a\xi^a{}_{,T}\right) + O(3)\,, \tag{2.19}$$

$$\delta U^a = \xi^a{}_{,T}\left(1+\frac{\widetilde{W}_G}{c^2}\right)+O(4)\,, \tag{2.20}$$

$$\delta U_0 = \frac{1}{c}\delta\widetilde{W}+O(3)\,, \tag{2.21}$$

$$\delta U_a = D_{ab}\left\{\xi^b{}_{,T}\left(1+\frac{\widetilde{W}_G}{c^2}\right)+\frac{1}{c^2}\left[2\widetilde{W}\xi^b{}_{,T}-4\delta\widetilde{W}^b+\widetilde{V}^b(\delta\widetilde{W}+\widetilde{V}_a\xi^a{}_{,T})\right]\right\}+O(4) \tag{2.22}$$

and

$$\delta\gamma^0{}_0 = -\frac{1}{c^2}\widetilde{V}_a\xi^a_{,T}+O(4)\,, \tag{2.23}$$

$$\delta\gamma^0{}_b = \frac{1}{c}D_{bc}\xi^c{}_{,T}+O(3)\,, \tag{2.24}$$

$$\delta\gamma^a{}_0 = -\frac{1}{c}\xi^a{}_{,T}+O(3)\,, \tag{2.25}$$

$$\delta\gamma^a{}_b = \frac{1}{c^2}\xi^a{}_{,T}D_{bc}\widetilde{V}^c+O(4)\,. \tag{2.26}$$

2. Lagrangian strain tensor and shear tensor

For a perfect elastic body, the energy-momentum tensor of the relaxed state is of the form

$$T_{\alpha\beta}=\rho U_\alpha U_\beta+p\gamma_{\alpha\beta}\,, \tag{2.27}$$

where ρ is the rest energy density and p the isotropic pressure in the reference state. The perturbed configuration changes the energy-momentum distribution and geometrical shape with time and might experience tidal forces from other astronomical objects. The Eulerian variation of the energy-momentum tensor can be expressed as

$$\delta T_{\alpha\beta}=U_\alpha U_\beta\delta\rho+\rho\delta(U_\alpha U_\beta)+\gamma_{\alpha\beta}\delta p+p\delta\gamma_{\alpha\beta}-2\mu s_{\alpha\beta}\,, \tag{2.28}$$

where $s_{\alpha\beta}$ is the shear tensor and μ is the shear modulus. The symmetric trace-free shear tensor is defined as

$$s_{\mu\nu}\equiv e_{\mu\nu}-\frac{1}{3}\Theta\gamma_{\mu\nu}\,, \tag{2.29}$$

where the volume dilatation is given by $\Theta=e^\mu{}_\mu$.
The Lagrangian strain tensor is defined by

$$e_{\mu\nu}=\frac{1}{2}\Delta\gamma_{\mu\nu}=\frac{1}{2}(\gamma_{\mu\nu}-\gamma^*_{\mu\nu})\,, \tag{2.30}$$

where $\gamma^*_{\mu\nu}$ is the unstrained value. Restricting ourselves to linear perturbations, the strain tensor is given by [15]

$$e_{\mu\nu}=\frac{1}{2}\gamma^\alpha_\mu\gamma^\beta_\nu\left(h_{\alpha\beta}+2\xi_{(\alpha;\beta)}\right)\,. \tag{2.31}$$

In the following we choose the displacement field in rotating spherical coordinates as $\xi^\beta=(0,\xi^r,\xi^\theta,\xi^\phi)$ and we use the linear displacement

$$\mathbf{D}\equiv D^r\mathbf{e}_r+D^\theta\mathbf{e}_\theta+D^\phi\mathbf{e}_\phi \tag{2.32}$$

with

$$D^r=\xi^r,\qquad D^\theta=r\xi^\theta,\qquad D^\phi=r\sin\theta\xi^\phi\,. \tag{2.33}$$

The explicit calculation of the Lagrangian strain tensor $e_{\mu\nu}$ and the shear tensor $s_{\mu\nu}$ is straightforward but cumbersome. Results for $e_{\mu\nu}$ and $s_{\mu\nu}$ to post-Newtonian accuracy are given in the Appendix.

3. The post-Newtonian energy and Eulerian equations

The Eulerian variation of the pressure δp can be derived similarly as in our previous paper dealing with Cartesian coordinates [10]. In spherical coordinates it takes the form

$$\delta p = -\rho^* \mathbf{D} \cdot \nabla \widetilde{W}_G - \kappa \Theta + \frac{\kappa}{c^2} \left(4\Theta \widetilde{W} + \Omega^2 r \sin\theta (D^\phi_{,\phi} + \sin\theta D^r + \cos\theta D^\theta) \right) + O(4) \, . \tag{2.34}$$

Our main results concern the perturbed local evolution equations

$$\delta \left(\widetilde{T}^\nu{}_{\mu;\nu} \right) = 0 \, . \tag{2.35}$$

For $\mu = 0$ one derives the perturbed energy equation in the form

$$\begin{aligned} \delta\rho &= -\mathbf{D} \cdot \nabla \rho - \rho \Theta - \frac{1}{c^2} p \Theta \\ &= -\left(D^r \rho_{,r} + \frac{1}{r} D^\theta \rho_{,\theta} + \frac{1}{r \sin\theta} D^\phi \rho_{,\phi} \right) - \rho^* \Theta + O(4) \, . \end{aligned} \tag{2.36}$$

The perturbed Euler equations correspond to the case $\mu = a$. As an intermediate result by using the expression for $\nabla \cdot \mathbf{D}$ from the Appendix we get

$$\begin{aligned} \nabla^2 \mathbf{D} &= \mathbf{e}_r \left[\nabla^2 D^r - \frac{2}{r^2} D^r - \frac{2}{r^2 \sin\theta} (\sin\theta D^\theta)_{,\theta} - \frac{2}{r^2 \sin\theta} (D^\phi)_{,\phi} \right] \\ &+ \mathbf{e}_\theta \left[\nabla^2 D^\theta - \frac{D^\theta}{r^2 \sin^2\theta} + \frac{2}{r^2} (D^r)_{,\theta} - \frac{2\cos\theta}{r^2 \sin^2\theta} (D^\phi)_{,\phi} \right] \\ &+ \mathbf{e}_\phi \left[\nabla^2 D^\phi - \frac{D^\phi}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin\theta} (D^r)_{,\phi} + \frac{2\cos\theta}{r^2 \sin^2\theta} (D^\theta)_{,\phi} \right] \, , \end{aligned}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors in the r, θ and ϕ direction respectively.

$$\begin{aligned} (\nabla \cdot \mathbf{D})_{,r} &= D^r_{,rr} - \frac{1}{r^2} D^\theta_{,\theta} + \frac{1}{r} D^\theta_{,\theta r} + \frac{1}{r \sin\theta} D^\phi_{,\phi r} - \frac{1}{r^2 \sin\theta} D^\phi_{,\phi} \\ &\quad + \frac{2}{r} D^r_{,r} - \frac{2}{r^2} D^r + \frac{1}{r} \cot\theta D^\theta_{,r} - \frac{1}{r^2} \cot\theta D^\theta \, , \\ (\nabla \cdot \mathbf{D})_{,\theta} &= D^r_{,r\theta} + \frac{1}{r} D^\theta_{,\theta\theta} + \frac{1}{r \sin\theta} D^\phi_{,\phi\theta} - \frac{\cos\theta}{r \sin^2\theta} D^\phi_{,\phi} + \frac{2}{r} D^r_{,\theta} \\ &\quad + \frac{1}{r} \cot\theta D^\theta_{,\theta} - \frac{1}{r \sin^2\theta} D^\theta \, , \\ (\nabla \cdot \mathbf{D})_{,\phi} &= D^r_{,r\phi} + \frac{1}{r} D^\theta_{,\theta\phi} + \frac{1}{r \sin\theta} D^\phi_{,\phi\phi} + \frac{2}{r} D^r_{,\phi} + \frac{1}{r} \cot\theta D^\theta_{,\phi} \, . \end{aligned}$$

Tedious calculations then lead to the explicit equations for $\mu = a = r$, θ, ϕ respectively.

The $\mu = r$ equation reads

$$\begin{aligned} 0 &= \rho^* \left(1 + \frac{2\widetilde{W}_G}{c^2} \right) \left(D^r_{,TT} - 2\Omega \sin\theta D^\phi_{,T} \right) + \rho^* \Theta \widetilde{W}_{G,r} - \rho^* (\delta \widetilde{W}_G)_{,r} \\ &\quad - \rho^* (\mathbf{D} \cdot \nabla \widetilde{W}_G)_{,r} - (\kappa\Theta)_{,r} - 2(\mu s^\beta{}_r)_{;\beta} + \frac{1}{c^2} \left\{ \rho^* \left[2\widetilde{W} D^r_{,TT} + \frac{8}{r} D^\theta_{,T} \widetilde{W}_{[\theta,r]} \right. \right. \\ &\quad \left. + \frac{8}{r \sin\theta} D^\phi_{,T} \widetilde{W}_{[\phi,r]} - 2\Omega r \sin\theta \widetilde{W}_{G,r} D^\phi_{,T} - 4(\delta \widetilde{W}^r)_{,T} \right] + \kappa \Theta \widetilde{W}_{G,r} \\ &\quad \left. + 4(\kappa \Theta \widetilde{W})_{,r} + \left[\kappa \Omega^2 \left(s_{\phi\phi} + \frac{1}{3} \Theta r^2 \sin^2\theta \right) \right]_{,r} \right\} + O(4) \, , \end{aligned} \tag{2.37}$$

where

$$
\begin{aligned}
2\left(\mu s^{\beta}{}_{r}\right)_{;\beta} &= 2\mu_{,\beta} s^{\beta}{}_{r} + 2\mu s^{\beta}{}_{r;\beta} \\
&= \mu\left(\frac{1}{3}(\nabla\cdot\mathbf{D})_{,r} + (\nabla^2\mathbf{D})_r\right) + 2\mu_{,r}\left(D^r_{,r} - \frac{1}{3}\nabla\cdot\mathbf{D}\right) \\
&+ \frac{\mu_{,\theta}}{r^2}\left(D^r_{,\theta} + rD^\theta_{,r} - D^\theta\right) + \frac{\mu_{,\phi}}{r\sin\theta}\left(\frac{D^r_{,\phi}}{r\sin\theta} + D^\phi_{,r} - \frac{D^\phi}{r}\right) \\
&+ \frac{1}{c^2}\left\{2\mu\left[2s_{rr}\widetilde{W}_{,r} + \frac{2s_{r\theta}}{r^2}\widetilde{W}_{,\theta} + \frac{2s_{r\phi}}{r^2\sin^2\theta}\widetilde{W}_{,\phi}\right.\right. \\
&+ 2\Omega s_{r\phi,T} - \frac{5}{6}\Omega\sin\theta(rD^\phi_{,T})_{,r} - \frac{1}{3}\Omega^2 r\sin\theta\cos\theta D^\theta_{,r} - \frac{\Omega^2}{2}D^r_{,\phi\phi} \\
&\left. - \frac{7}{3}\Omega^2\sin^2\theta(D^r + D^\theta\cot\theta) - \Omega^2\sin\theta D^\phi_{,\phi}\right] \\
&\left. - \frac{2}{3}\mu_{,r}\Omega r\sin\theta\left(D^\phi_{,T} + D^r\Omega\sin\theta + D^\theta\Omega\cos\theta\right) + \mu_{,\phi}\Omega\left(D^r_{,T} - \Omega D^r_{,\phi}\right)\right\} + O(4)\,.
\end{aligned}
\tag{2.38}
$$

Two indices enclosed in parentheses imply symmetrization as in $\widetilde{W}_{(a,b)} = (\widetilde{W}_{a,b} + \widetilde{W}_{b,a})/2$, and two indices enclosed in square brackets imply anti-symmetrization as in $\widetilde{W}_{[a,b]} = (\widetilde{W}_{a,b} - \widetilde{W}_{b,a})/2$.

The $\mu = \theta$ equation takes the form

$$
\begin{aligned}
0 &= \rho^* r\left(1 + \frac{2\widetilde{W}_G}{c^2}\right)\left(D^\theta_{,TTt} - 2\Omega\cos\theta D^\phi_{,T}\right) + \rho^*\Theta\widetilde{W}_{G,\theta} - \rho^*(\delta\widetilde{W}_G)_{,\theta} \\
&- \rho^*(\mathbf{D}\cdot\nabla\widetilde{W}_G)_{,\theta} - (\kappa\Theta)_{,\theta} - 2(\mu s^\beta_\theta)_{;\beta} + \frac{1}{c^2}\left\{\rho^*\left[2r\widetilde{W}D^\theta_{,TT} + 8D^r_{,T}\widetilde{W}_{[r,\theta]}\right.\right. \\
&\left. + \frac{8}{r\sin\theta}D^\phi_{,T}\widetilde{W}_{[\phi,\theta]} - 2\Omega r\sin\theta\widetilde{W}_{G,\theta}D^\phi_{,T} - 4(\delta\widetilde{W}_\theta)_{,T}\right] + \kappa\Theta\widetilde{W}_{G,\theta} \\
&\left. + 4(\kappa\Theta\widetilde{W})_{,\theta} + \left[\kappa\Omega^2\left(s_{\phi\phi} + \frac{1}{3}\Theta r^2\sin^2\theta\right)\right]_{,\theta}\right\} + O(4)\,,
\end{aligned}
\tag{2.39}
$$

where

$$
\begin{aligned}
2\left(\mu s^{\beta}{}_{\theta}\right)_{;\beta} &= 2\mu_{,\beta} s^{\beta}{}_{\theta} + 2\mu s^{\beta}{}_{\theta;\beta} \\
&= \mu\left(\frac{1}{3}(\nabla\cdot\mathbf{D})_{,\theta} + r(\nabla^2\mathbf{D})_\theta\right) + \mu_{,r}\left(D^r_{,\theta} + rD^\theta_{,r} - D^\theta\right) \\
&+ 2\mu_{,\theta}\left(\frac{D^\theta_{,\theta}}{r} + \frac{D^r}{r} - \frac{1}{3}\nabla\cdot\mathbf{D}\right) + \frac{\mu_{,\phi}}{r\sin\theta}\left(\frac{D^\theta_{,\phi}}{\sin\theta} + D^\phi_{,\theta} - D^\phi\cot\theta\right) \\
&+ \frac{1}{c^2}\left\{2\mu\left[2s_{\theta r}\widetilde{W}_{,r} + \frac{2s_{\theta\theta}}{r^2}\widetilde{W}_{,\theta} + \frac{2s_{\theta\phi}}{r^2\sin^2\theta}\widetilde{W}_{,\phi} + 2\Omega s_{\theta\phi,T} - \frac{5}{6}\Omega r(\sin\theta D^\phi_{,T})_{,\theta}\right.\right. \\
&- \frac{1}{2}\Omega^2 rD^\theta_{,\phi\phi} + \frac{1}{3}\Omega^2 r^2\sin\theta\cos\theta D^r_{,r} - \frac{1}{3}\Omega^2 r\sin^2\theta D^r_{,\theta} - \frac{2}{3}\Omega^2 r\cos\theta D^\phi_{,\phi} \\
&\left. - 2\Omega^2 r\sin\theta\cos\theta D^r + \frac{1}{3}\Omega^2 r^2\sin^2\theta D^\theta - 2\Omega^2 r\cos^2\theta D^\theta\right] \\
&- \frac{2}{3}\mu_{,\theta}\Omega r\sin\theta\left(D^\phi_{,T} + \Omega(D^r\sin\theta + D^\theta\cos\theta)\right) \\
&\left. + \mu_{,\phi}\Omega r\left(D^\theta_{,T} - \Omega D^\theta_{,\phi}\right)\right\} + O(4)\,.
\end{aligned}
\tag{2.40}
$$

Finally, the $\mu = \phi$ equation reads explicitly

$$
0 = \rho^* r\sin\theta\left(1 + \frac{2\widetilde{W}_G}{c^2}\right)\left(D^\phi_{,TT} + 2\Omega\sin\theta(D^r_{,T} + \cot\theta D^\theta_{,T})\right)
$$

$$
\begin{aligned}
&+\rho^*\Theta\widetilde{W}_{G,\phi} - \rho^*(\delta\widetilde{W}_G)_{,\phi} - \rho^*(\mathbf{D}\cdot\nabla\widetilde{W}_G)_{,\phi} - (\kappa\Theta)_{,\phi} - 2(\mu s^\beta{}_\phi)_{;\beta} \\
&+\frac{1}{c^2}\left\{\rho^*\left[2r\sin\theta\widetilde{W}_G D^\phi_{,TT} + \Omega r^2\sin^2\theta(\mathbf{D}_{,T}\cdot\nabla\widetilde{W}_G) - 2\Omega r\sin\theta\widetilde{W}_{G,\phi}D^\phi_{,T}\right.\right. \\
&\left.+8D^r_{,T}\widetilde{W}_{[r,\phi]} + \frac{8}{r}D^\theta_{,T}\widetilde{W}_{[\theta,\phi]} + \Omega r^2\sin^2\theta(\delta\widetilde{W})_{,T} - 4(\delta\widetilde{W}_\phi)_{,T}\right] \\
&\left.+\kappa(\Theta\widetilde{W}_{G,\phi} - \Theta_{,T}\Omega r^2\sin^2\theta) + 4(\kappa\Theta\widetilde{W})_{,\phi} + \left[\kappa\Omega^2\left(s_{\phi\phi} + \frac{1}{3}\Theta r^2\sin^2\theta\right)\right]_{,\phi}\right\} + O(4)\,,
\end{aligned}
\tag{2.41}
$$

where

$$
\begin{aligned}
2\left(\mu s^\beta{}_\phi\right)_{;\beta} &= 2\mu_{,\beta}s^\beta{}_\phi + 2\mu s^\beta{}_{\phi;\beta} \\
&= \mu\left(\frac{1}{3}(\nabla\cdot\mathbf{D})_{,\phi} + r\sin\theta(\nabla^2\mathbf{D})_\phi\right) + \mu_{,r}r\sin\theta\left(\frac{D^r_{,\phi}}{r\sin\theta} + D^\phi_{,r} - \frac{D^\phi}{r}\right) \\
&\quad+\mu_{,\theta}\sin\theta\left(\frac{D^\phi_{,\theta}}{r} + \frac{D^\theta_{,\theta}}{r\sin\theta} - \frac{D^\phi\cot\theta}{r}\right) \\
&\quad+2\mu_{,\phi}\left(\frac{D^\phi_{,\phi}}{r\sin\theta} + \frac{D^r}{r} + \frac{D^\theta\cot\theta}{r} - \frac{1}{3}\nabla\cdot\mathbf{D}\right) \\
&\quad+\frac{1}{c^2}\left\{2\mu\left[2s_{\phi r}\widetilde{W}_{,r} + \frac{2s_{\phi\theta}}{r^2}\widetilde{W}_{,\theta} + \frac{2s_{\phi\phi}}{r^2\sin^2\theta}\widetilde{W}_{,\phi} + \frac{1}{6}\Omega r^2\sin^2\theta(\nabla\cdot\mathbf{D})_{,T}\right.\right. \\
&\quad+\frac{1}{2}\Omega^2r^3\sin^3\theta(\nabla^2\mathbf{D})_\phi + \frac{7}{6}\Omega r\sin\theta D^\phi_{,\phi t} \\
&\quad+2\Omega r\sin\theta(D^r_{,T}\sin\theta + D^\theta_{,T}\cos\theta) - \frac{1}{3}\Omega^2r\sin\theta\left(D^r_{,\phi}\sin\theta + D^\theta_{,\phi}\cos\theta\right) \\
&\quad\left.+\frac{1}{2}\Omega^2r\sin\theta\left(r\sin^2\theta D^\phi_{,r} - D^\phi + 2\cos\theta D^\phi_{,\theta} - D^\phi_{,\phi\phi}\right)\right] \\
&\quad+\mu_{,r}\Omega r^2\sin^2\theta\left[\Omega r\sin\theta\left(\frac{D^r_{,\phi}}{r\sin\theta} + D^\phi_{,r} - \frac{D^\phi}{r}\right) + D^r_{,T} - \Omega D^r_{,\phi}\right] \\
&\quad+\mu_{,\theta}\Omega r\sin^2\theta\left[\Omega r\sin\theta\left(\frac{D^\phi_{,\theta}}{r} + \frac{D^\theta_{,\theta}}{r\sin\theta} - \frac{D^\phi\cot\theta}{r}\right) + D^\theta_{,T} - \Omega D^\theta_{,\phi}\right] \\
&\quad\left.+\frac{4}{3}\mu_{,\phi}\Omega r\sin\theta\left(D^\phi_{,T} + \Omega(D^r\sin\theta + D^\theta\cos\theta)\right)\right\} + O(4)\,.
\end{aligned}
\tag{2.42}
$$

Eqs.(2.37), (2.39), (2.41) together with (2.36) are the desired dynamical equations for the displacement field. They are valid up to terms of order $1/c^4$ and second order in the displacements field itself. The Newtonian limit of our results (neglecting all $1/c^2$ terms) agrees with standard results from textbooks (e.g., the ones from [17] after correction of a typographical mistake).

III. POST-NEWTONIAN JUNCTION CONDITIONS

In most cases such post-Newtonian dynamical equations of elastic deformable bodies will be applied to astronomical bodies composed of different layers. If the body has several layers as e.g., the Earth that shows a solid inner core, a fluid outer core, a mantle and a thin crust, we have to consider corresponding junction conditions at the interface of two adjacent layers. First we consider such junction conditions in Cartesian coordinates. For practical applications they are then also formulated in spherical coordinates where they can be compared with well-known Newtonian results [3]. Junctions conditions are formulated in rotating coordinates and to have a well defined stationary and axisymmetric ground state it is assumed that all layers rotate with the same angular velocity $\mathbf{\Omega}$, i.e. there is no relative motion between two layers in the ground state. The behavior of individual physical quantities and their corresponding perturbed quantities on the interface will be studied at first.

The gravitational potentials W and W^a are physical quantities in the non-rotating ground state. For a isolated body, W and W^a can be obtained as a solution of Eqs.(2.4), (2.5) in [10]. They are inhomogeneous D'Alembert's and Poisson differential equation respectively. Although the sources Σ and Σ^a may be discontinuous across any interface, W and W^a (solutions of the equations) are continue on the interface. Since we do not consider the shock wave, the surface mass-density and surface current mass-density do not exist anywhere. Therefore derivatives of W and W^a have to be continue on the interface also. $\overline{W}$ and $\overline{W}^a$ in rotating coordinates differ from (W, W^a) simply by a nonsingular coordinate transformation (Eq.(2.4) and (2.5)). So $\overline{W}$, $\overline{W}^a$ and their derivative continuous on any interface as well.

To discuss the continuity of $\delta\overline{W}$ and $\delta\overline{W}^a$ we have to consider δW and δW^a in non-rotating coordinates before hand, since only in non-rotating coordinates δW and δW^a can be obtained from perturbed field equations (Eq.(4.17) and (4.18) of Ref.[10])

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial T^2}\right)\delta W = -4\pi G\delta\Sigma + O(4) \tag{3.1}$$

$$\nabla^2\delta W^a = -4\pi G\delta\Sigma^a + O(2) \tag{3.2}$$

where

$$\begin{aligned}\delta\Sigma &= -\rho^*_{,a}s^a - \rho^*\Theta + \frac{1}{c^2}\Big[2\rho^*(\delta\overline{W} + 2\overline{V}^a\dot{s}^a) - 2(\rho^*_{,a}s^a + \rho^*\Theta)(\overline{W} + \overline{V}^2) \\ &\quad -2\rho^*\overline{W}_{\mathrm{G},b}s^b - 3\kappa\Theta\Big] + O(4)\,,\end{aligned} \tag{3.3}$$

$$\delta\Sigma^a = R^{ba}\left[\rho^*\dot{s}^b - \overline{V}^b(\rho^*_{,c}s^c + \rho^*\Theta)\right] + O(2)\,. \tag{3.4}$$

δW and δW^a are solutions of Eqs.(3.1), (3.2). They are continuous as the same discussion on W and W^a, but their derivatives are in different cases. The Eulerian variation of the surface mass-density and Eulerian variation of the surface current mass-density do exist on the interface because both $\delta\Sigma$ and $\delta\Sigma^a$ are dependent on the spatial coordinate derivatives of ρ (to see Eq.(3.3), (3.4)). Then on interface $\delta\Sigma$ and $\delta\Sigma^a$ are divergence. Therefore $(\delta W)_{,a}$ and $(\delta W^a)_{,b}$ are finite on the interface but not necessary to be continuous across the interface. Through the coordinate transformation and neglecting all of higher order terms, we get the representation of $\delta\overline{W}$ and $\delta\overline{W}^a$ by means of δW and δW^a (see (4.22) and (4.23) of [10])

$$\delta\overline{W} = \delta W + \frac{1}{c^2}\left(2\delta W V^2 - 4\delta W^b V^b\right) + O(4)\,, \tag{3.5}$$

$$\delta\overline{W}^a = R^{ab}\left(\delta W^b - \frac{1}{2}V^b\delta W\right) + O(2)\,. \tag{3.6}$$

The inverse transformation takes the form:

$$\delta W = \delta\overline{W} + \frac{4}{c^2}\overline{V}^a\delta\overline{W}^a + O(4)\,, \tag{3.7}$$

$$\delta W^a = R^{ba}\left(\delta\overline{W}^b + \frac{1}{2}\overline{V}^b\delta\overline{W}\right) + O(2)\,. \tag{3.8}$$

The transformation formulae is nonsingular, then the behavior of $\delta\overline{W}$ and $\delta\overline{W}^a$ are similar to δW and δW^a, i.e. $\delta\overline{W}$ and $\delta\overline{W}^a$ are continuous across any interface. $\delta\overline{W}_{,a}$ and $(\delta\overline{W}^a)_{,b}$ are finite on the interface, but not necessary continuous. The first and second time derivatives of $\delta\overline{W}$ and $\delta\overline{W}^a$ are continuous everywhere including in the interface since the shock wave does not exist at anytime. Later when we consider junction condition, we can drop the continuous terms on both sides of the interface.

For the displacement field $\mathbf{s}$ we shall take a similar physical consideration as in [3]. The field $\mathbf{s}$ is continuous across any solid-solid interface. Its normal component $\mathbf{n}\cdot\mathbf{s}$ is continuous across any interface. But its tangent component maybe not continuous for solid-liquid interface, since the tangent interaction between solid and liquid is close to zero in the absence of viscous forces. Also $\dot{s}^a$ and $\ddot{s}^a$ are finite across any interface. μ, κ, s_{ab} and Θ are finite, but on the different side of a interface they maybe different (not continuous). Therefore $\mu_{,a}$, $\kappa_{,a}$, $s_{bc,a}$ and $\Theta_{,a}$ are not necessary finite on the interface. As we mentioned before all of layers rotate with the same angular velocity $\mathbf{\Omega}$, so that V^a and $\overline{V}^a(\overline{V}^a = R^{ab}V^b)$ are continuous across the interface. $V^b_{,a}$, $\overline{V}^b_{,a}$ and $V^2_{,a} = \overline{V}^2_{,a}$ are continuous as well.

In Eq.(3.1) the D'Alembertian can be related to the non-rotating coordinates. A transformation to rotating coordinates yields

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial T^2} = \overline{\nabla}^2 + \frac{2\overline{V}^a}{c^2}\frac{\partial^2}{\partial\overline{X}^a\partial\overline{T}} - \frac{\overline{V}^a\overline{V}^b}{c^2}\frac{\partial^2}{\partial\overline{X}^b\partial\overline{X}^a} - \frac{1}{c^2}\frac{\partial^2}{\partial\overline{T}^2}\,, \tag{3.9}$$

where we have used the relation $\partial\overline{V}^a/\partial\overline{X}^a = 0$. Substituting Eq.(3.9) and (3.7) into Eq.(3.1), we get

$$\frac{\partial}{\partial\overline{X}^a}\left[\frac{\partial}{\partial\overline{X}^a}\delta\overline{W} + \frac{2}{c^2}\overline{V}^a\delta\overline{W}_{,\overline{T}} + \frac{4}{c^2}\frac{\partial}{\partial\overline{X}^a}(\overline{V}^b\delta\overline{W}^b) - \frac{\overline{V}^a\overline{V}^b}{c^2}\frac{\partial\delta\overline{W}}{\partial\overline{X}^b}\right] - \frac{1}{c^2}\delta\overline{W}_{,\overline{TT}}$$
$$= -4\pi G\delta\Sigma + O(4)\,, \tag{3.10}$$

We also can rewrite $\delta\Sigma$ (to see Eqs.(4.27), (4.31) and (4.40) of [10])

$$\begin{aligned}\delta\Sigma \;=\; & -\overline{\nabla}\cdot(\rho\mathbf{s}) + \frac{1}{c^2}\Big[3\rho\overline{V}^a\dot{s}^a - (ps^a)_{,a} - 3\rho\overline{W}_{,b}s^b - \rho\delta\overline{W} \\ & -2\overline{\nabla}\cdot\Big(\rho\mathbf{s}(\overline{W}+\overline{V}^2)\Big) + \frac{1}{2}\rho\overline{V}^2_{\;,b}s^b - 3\kappa s^b_{\;,b}\Big]\,. \end{aligned} \tag{3.11}$$

From now on for convenience we omitted "bar" in $\overline{\nabla}$ as we did in [10]. Substituting Eq.(3.11) into Eq.(3.10), we get

$$\nabla\cdot\mathbf{A} + B = O(4) \tag{3.12}$$

with

$$\begin{aligned}\mathbf{A} \;=\; & \nabla\delta\overline{W} - 4\pi G\rho^*\mathbf{s} \\ & +\frac{1}{c^2}\Big[2\overline{\mathbf{V}}\delta\overline{W}_{,\overline{T}} + 4\nabla(\overline{V}^b\delta\overline{W}^b) - \overline{\mathbf{V}}(\overline{\mathbf{V}}\cdot\nabla\delta\overline{W}) - 8\pi G\rho^*\mathbf{s}(\overline{W}+\overline{V}^2)\Big]\end{aligned}$$

and

$$B = \frac{1}{c^2}\Big(12\pi G\rho\mathbf{s}\cdot\overline{\mathbf{V}} - 12\pi G\rho\mathbf{s}\cdot\nabla\overline{W} - 4\pi G\rho\delta\overline{W} + 2\pi G\rho\mathbf{s}\cdot\nabla\overline{V}^2 - 12\pi G\kappa\Theta - \delta\overline{W}_{,\overline{TT}}\Big)\,.$$

As the discussion on the boundary condition problem in classical physics, we now integrate Eq.(3.12) over an infinitesimally small volume ΔV such that an interface intersects this volume. We choose ΔV as a cylinder. The surface (S) of ΔV enclose the interface of two different layers with the shape of a circular drum of a radius r and a depth h, where $h \ll r$ and r is so small that the portion of the interface contained in (S) can be taken as flat. When $h \to 0$ the integration divide into two parts: the first part of Eq.(3.12) becomes a surface integration by means of Gauss' theorem, the second part tends to zero with vanishing ΔV because the terms in B are all continuous or finite. Hence,

$$\mathbf{n}\cdot\mathbf{A}|_{\rm layer1} = \mathbf{n}\cdot\mathbf{A}|_{\rm layer2}\,, \tag{3.13}$$

i.e.

$$\mathbf{n}\cdot\left\{\nabla\delta\overline{W} - 4\pi G\rho^*\mathbf{s} + \frac{1}{c^2}\Big[4\nabla(\overline{V}^b\delta\overline{W}_b) - \overline{\mathbf{V}}(\overline{\mathbf{V}}\cdot\nabla\delta\overline{W}) - 8\pi G\rho^*\mathbf{s}(\overline{W}+\overline{V}^2)\Big]\right\} + O(4)$$
$$\text{continuous across any interface}\,, \tag{3.14}$$

here we have dropped the term $2\overline{\mathbf{V}}(\delta\overline{W})_{,T}$ since it is continuous across interface.

Substituting Eq.(3.8) and (3.4) into (3.2) and considering $\overline{\nabla}^2 = \nabla^2$ and the rotation matrix R^{ba} dependent on time only, we get

$$\frac{\partial}{\partial\overline{X}^b}\left[\frac{\partial}{\partial\overline{X}^b}\left(\delta\overline{W}^a + \frac{1}{2}\overline{V}^a\delta\overline{W}\right) - 4\pi G s^b\rho\overline{V}^a\right] + 4\pi G\left(\rho\dot{s}^a + \rho s^b\frac{\partial\overline{V}^a}{\partial\overline{X}^b}\right) = O(2)\,. \tag{3.15}$$

Performing a similar integration over an infinitesimal volume as before, we find that

$$\mathbf{n}\cdot\left[\nabla(\delta\overline{W}^a + \frac{1}{2}\overline{V}^a\delta\overline{W}) - 4\pi G\rho^*\mathbf{s}\overline{V}^a\right] + O(2) \qquad \text{continuous across any interface}\,, \tag{3.16}$$

where ρ has been substituted by ρ^* since this equation is valid only to $O(2)$. Eq.(3.16) can be written by means of the form of parallel vector, i.e.

$$\mathbf{n}\cdot\left[\nabla(\delta\overline{\mathbf{W}} + \frac{1}{2}\overline{\mathbf{V}}\delta\overline{W}) - 4\pi G\rho^*\mathbf{s}\overline{\mathbf{V}}\right] + O(2) \qquad \text{continuous across any interface}\,. \tag{3.17}$$

Finally a similar integration of Eq.(2.8) over an infinitesimal cylindrical volume on the interface leads to

$$\int_{\Delta V} (A_{ba,b} + B_a) dV = 0\,, \tag{3.18}$$

where

$$\begin{aligned}
A_{ba} &= -\kappa\Theta\delta_{ab} - 2\mu s_{ba} + \frac{1}{c^2}\left(4\mu\overline{W}s_{ba} + 2\mu s_{ca}\overline{V}^c\overline{V}^b + 4\kappa\overline{W}\Theta\delta_{ab} + \kappa\overline{V}^d\overline{V}^c s^d_{,c}\delta_{ab}\right)\,, \qquad (3.19)\\
B_a &= \rho^*\left(1 + \frac{2\overline{W}_{\mathrm{G}}}{c^2}\right)\left(\ddot{s}_a + 2\epsilon_{abc}\overline{\Omega}^b\dot{s}^c\right) + \rho^*\Theta\overline{W}_{\mathrm{G},a} - \rho^* s^b_{,a}\overline{W}_{\mathrm{G},b} - \rho^*(\delta\overline{W}_{\mathrm{G}})_{,a} - \rho^* s^b\overline{W}_{\mathrm{G},ba}\\
&\quad + \frac{1}{c^2}\left\{\rho^*\left[\overline{V}^a(\overline{V}^b\ddot{s}^b) + \overline{W}_{\mathrm{G},b}\dot{s}^b\overline{V}^a - 2\overline{V}^b\dot{s}^b\overline{W}_{\mathrm{G},a} + (\delta\overline{W})_{,\overline{T}}\overline{V}^a + 8\dot{s}^b\overline{W}_{[b,a]} - 4(\delta\overline{W}_a)_{,\overline{T}}\right]\right.\\
&\quad \left. -(2\mu s_{ab})_{,\overline{T}}\overline{V}^b - 2\mu(2\overline{W}_{,c}s_{ac} + \epsilon_{acb}\Omega^c s_{bd}\overline{V}^d) + \kappa(\Theta\overline{W}_{\mathrm{G},a} - \dot{\Theta}\overline{V}^a)\right\}\,. \qquad (3.20)
\end{aligned}$$

All of terms in B_a are finite or continuous as we mentioned before. In terms of median method, we have

$$\int_{\Delta V} B_a dV = \overline{B}_a\pi r^2 h\,. \tag{3.21}$$

The first term of Eq.(3.18) can be deduced as an surface integration

$$\int_{\Delta V} A_{ba,b} dV = \pi r^2(n_1^b A_{ab} - n_2^b A_{ab})\,. \tag{3.22}$$

Cancelling πr^2 and neglecting the higher-order term $\overline{B}_a h$, we get

$$\begin{aligned}
&n^b\left\{\kappa\Theta\delta_{ab} + 2\mu s_{ab} - \frac{1}{c^2}\left[4\mu\overline{W}s_{ba} + 2\mu s_{ca}\overline{V}^c\overline{V}^b + 4\kappa\overline{W}\Theta\delta_{ab} + \kappa\overline{V}^d\overline{V}^c s^d{}_{,c}\delta_{ab}\right]\right\} + O(4)\\
&\qquad \text{continuous across any interface}\,. \qquad (3.23)
\end{aligned}$$

We should point out that the Newtonian part of Eq.(3.23) $\kappa\Theta\delta_{ab} + 2\mu s_{ab}$ is just the Newtonian stress tensor T_{ab} in [3], therefore Eq.(3.23) is an extended PN version of Wahr's.

The PN junction conditions in Cartesian coordinates are summarized as follows:

$$\begin{aligned}
&\mathbf{s} \quad \text{continuous across any solid} - \text{solid interface} &(3.24)\\
&\mathbf{s}\cdot\mathbf{n} \quad \text{continuous across any interface} &(3.25)\\
&\delta\overline{W} \quad \text{continuous across any interface} &(3.26)\\
&\delta\overline{W}^a \quad \text{continuous across any interface} &(3.27)\\
&\mathbf{n}\cdot\left\{\nabla\delta\overline{W} - 4\pi G\rho^*\mathbf{s} + \frac{1}{c^2}\left[4\nabla\left(\overline{V}^b\delta\overline{W}_b\right) - \overline{\mathbf{V}}(\overline{\mathbf{V}}\cdot\nabla(\delta\overline{W})) - 8\pi G\rho^*\mathbf{s}(\overline{W} + \overline{V}^2)\right]\right\} + O(4)\\
&\qquad \text{continuous across any interface} &(3.28)\\
&\mathbf{n}\cdot\left[\nabla(\delta\overline{\mathbf{W}} + \frac{1}{2}\overline{\mathbf{V}}\delta\overline{W}) - 4\pi G\rho^*\mathbf{s}\overline{\mathbf{V}}\right] + O(2) \qquad \text{continuous across any interface} &(3.29)\\
&n^b\left\{\kappa\Theta\delta_{ab} + 2\mu s_{ab} - \frac{1}{c^2}\left[4\mu\overline{W}s_{ab} + 2\mu s_{ac}\overline{V}^c\overline{V}^b + 4\kappa\overline{W}\Theta\delta_{ab} + \kappa\delta_{ab}\overline{V}^d\overline{V}^c s^d{}_{,c}\right]\right\} + O(4)\\
&\qquad \text{continuous across any interface}\,. &(3.30)
\end{aligned}$$

For the outer surface boundary conditions we only need simply take $\mu = \kappa = \rho = \mathbf{s} = 0$ outside the elastic body, i.e.

$$\begin{aligned}
&\delta\overline{W}|_{\mathrm{in}} = \delta\overline{W}|_{\mathrm{out}} &(3.31)\\
&\delta\overline{W}^a|_{\mathrm{in}} = \delta\overline{W}^a|_{\mathrm{out}} &(3.32)\\
&\mathbf{n}\cdot\left\{\nabla\delta\overline{W} - 4\pi G\rho^*\mathbf{s} + \frac{1}{c^2}\left[4\nabla\left(\overline{V}^b\delta\overline{W}_b\right) - \overline{\mathbf{V}}(\overline{\mathbf{V}}\cdot\nabla(\delta\overline{W})) - 8\pi G\rho^*\mathbf{s}(\overline{W} + \overline{V}^2)\right]\right\}|_{\mathrm{in}}
\end{aligned}$$

$$= \mathbf{n}\cdot\left\{\nabla\delta\overline{W}+\frac{1}{c^2}\left[4\nabla\left(\overline{V}^b\delta\overline{W}_b\right)-\overline{\mathbf{V}}(\overline{\mathbf{V}}\cdot\nabla(\delta\overline{W}))\right]\right\}|_{\rm out}+O(4) \tag{3.33}$$

$$\mathbf{n}\cdot\left\{\nabla\left(\delta\overline{\mathbf{W}}+\frac{1}{2}\overline{\mathbf{V}}\delta\overline{W}\right)-4\pi G\rho^*\mathbf{s}\overline{\mathbf{V}}\right\}|_{in}=\mathbf{n}\cdot\nabla\left(\delta\overline{\mathbf{W}}+\frac{1}{2}\overline{\mathbf{V}}\delta\overline{W}\right)|_{out}+O(2) \tag{3.34}$$

$$n^b\left\{\kappa\Theta\delta_{ab}+2\mu s_{ab}-\frac{1}{c^2}\left[4\mu\overline{W}s_{ab}+2\mu s_{ac}\overline{V}^c\overline{V}_b+4\kappa\overline{W}\Theta\delta_{ab}+\kappa\delta_{ab}\overline{V}_d\overline{V}^c s^d{}_{,c}\right]\right\}=O(4)\,. \tag{3.35}$$

For most applications the use of spherical coordinates will be advantageous. Corresponding boundary and junction conditions cab be derived similarly to the case of Cartesian coordinates. They read

$$\mathbf{D}\quad \text{continuous across any solid} - \text{solid interface} \tag{3.36}$$

$$\mathbf{D}\cdot\mathbf{n}\quad \text{continuous across any interface} \tag{3.37}$$

$$\delta\widetilde{W}\quad \text{continuous across any interface} \tag{3.38}$$

$$\delta\widetilde{W}^a\quad \text{continuous across any interface } (a=r,\theta,\phi \text{ respectively}) \tag{3.39}$$

$$\mathbf{n}\cdot\left\{\nabla\delta\widetilde{W}-4\pi G\rho^*\mathbf{D}+\frac{1}{c^2}\left[4\nabla(\Omega r^2\sin^2\theta\delta\widetilde{W}^\phi)-8\pi G\rho^*\mathbf{D}(\widetilde{W}+\Omega^2r^2\sin^2\theta)\right.\right.$$
$$\left.\left.-\mathbf{e}_\phi\Omega^2 r\sin\theta(\delta\widetilde{W})_{,\phi}\right]\right\}+O(4)\qquad \text{continuous across any interface} \tag{3.40}$$

$$\mathbf{n}\cdot\left[\nabla\left(\delta\widetilde{\mathbf{W}}+\frac{1}{2}\widetilde{\mathbf{V}}\delta\widetilde{W}\right)-4\pi G\rho^*\mathbf{D}\widetilde{\mathbf{V}}\right]+O(2)\qquad \text{continuous across any interface} \tag{3.41}$$

$$n^b\left\{\kappa\Theta D_{ab}+2\mu s_{ab}-\frac{1}{c^2}\left[4\mu\widetilde{W}s_{ab}+2\mu s_{a\phi}\Omega^2 D_{b\phi}+4\kappa\widetilde{W}\Theta D_{ab}+\kappa D_{ab}\left(\Omega^2 r\sin\theta D^\phi{}_{,\phi}\right.\right.\right.$$
$$\left.\left.\left.+\frac{1}{2}\mathbf{D}\cdot\nabla(\Omega^2r^2\sin^2\theta)\right)\right]\right\}\qquad \text{continuous across any interface}\,. \tag{3.42}$$

s_{ab} in Eq.(3.42) is the shear tensor in spherical coordinates, which is shown in the Appendix Eqs.(A.10)–(A.15)

For the outer surface boundary we take $\mu=\kappa=\rho=\mathbf{D}=0$ outside the elastic body as before. When we neglect all of $1/c^2$ terms and let $\overline{V}^a=0$(non-rotating), our formulae (Eq.(3.40) and (3.42)) agree with Wahr's results (to see Sec. III of Ref.[3]), i.e. our work is an extension of the Newtonian version to rotating PN version. As for our formulae (Eq.(3.41)), since it is a purely PN junction condition, there is no Newtonian corresponding to be compared.

IV. CONCLUSION

In this paper we present the perturbation equations for the dynamical behavior of astronomical elastic bodies in the first post-Newtonian approximation of Einstein theory of gravity in rotating spherical coordinates. In comparison with our previous in Cartesian coordinates[10], these equations in spherical coordinates are more useful for applications since usually all relevant equations are expended in terms of scalar, vector and tensor spherical harmonics. The equations and relations such as junction conditions can e.g., be applied to problems of geodynamics (e.g., for the theory of nutation) or seismology of compact stars (e.g. for the problem of the normal modes of astronomical bodies). Also the formulation of post-Newtonian junction conditions at the interface of adjacent layers of different elastic properties is presented here for the first time. A comparison reveals that the Newtonian limit agrees with well-known results from the literature. Here it should be emphasized that the junction conditions can be written very generally with ordinary Euclidean 3-vectors and 3-tensors so that they can be formulated for a broad class of coordinate systems.

Our perturbation equations together with junctions conditions can in principle be solved if the internal quantities of state (density, pressure etc.) and elastic moduli are given e.g., by some Earth's model[20]. However, these equations describing the free and forced motions of the body are complicated partial differential equations. For that reason usually an expansion of relevant quantities in terms of spherical harmonics turns these equations into a set of coupled ordinary differential equations. A different natural basis for such an expansion is provided by the so-called generalized spherical harmonics[21, 22] that was employed by Wahr and other authors. By using the generalized spherical harmonics to expand all functions (displacement vector, incremental Eulerian gravitational potential energy, incremental elastic stress tensor, applied force (tidal force et al.)), the original partial differential equations and boundary conditions are transformed into a set of ordinary differential equations and scalar boundary conditions

for the unknowns functions. For a spherical and non-rotating ground state these ordinary differential equations are uncoupled, for an oblate, rotating body they are coupled, the coupling parameter being given by the dimensionless oblateness of the body. Such expansions together with applications for geodynamics will be published separately.

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APPENDIX: EXPLICIT RESULTS FOR THE STRAIN AND SHEAR TENSOR

From (2.31) one derives

$$
\begin{aligned}
e_{00} &= O(4)\,, && \text{(A.1)}\\
e_{0r} &= e_{0\theta} = e_{0\phi} = O(5)\,, && \text{(A.2)}\\
e_{rr} &= D^r_{,r} + \frac{1}{c^2}\left(\delta\widetilde{W} + \mathbf{D}\cdot\nabla\widetilde{W} + 2\widetilde{W}D^r_{,r}\right) + O(4)\,, && \text{(A.3)}\\
e_{\theta\theta} &= r^2\left\{\frac{1}{r}D^\theta_{,\theta} + \frac{1}{r}D^r + \frac{1}{c^2}\left(\delta\widetilde{W} + \frac{2\widetilde{W}}{r}D^r + \mathbf{D}\cdot\nabla\widetilde{W} + \frac{2\widetilde{W}}{r}D^\theta_{,\theta}\right)\right\} + O(4)\,, && \text{(A.4)}\\
e_{\phi\phi} &= r^2\sin^2\theta\left\{\frac{1}{r\sin\theta}D^\phi_{,\phi} + \frac{1}{r}D^r + \frac{1}{r}D^\theta\cot\theta + \frac{1}{c^2}\left[\delta\widetilde{W} + \mathbf{D}\cdot\nabla\widetilde{W}\right.\right. \\
&\quad \left.\left. + \frac{1}{r\sin\theta}(2\widetilde{W}+V^2)D^\phi_{,\phi} + \Omega r\sin\theta D^\phi_{,T} + \frac{2}{r}(\widetilde{W}+V^2)(D^r + D^\theta\cot\theta)\right]\right\} + O(4), && \text{(A.5)}\\
e_{r\theta} &= r\left\{\frac{1}{2r}(D^r_{,\theta} + rD^\theta_{,r} - D^\theta)\left(1 + \frac{2\widetilde{W}}{c^2}\right)\right\} + O(4)\,, && \text{(A.6)}\\
e_{r\phi} &= r\sin\theta\left\{\frac{1}{2}\left(\frac{D^r_{,\phi}}{r\sin\theta} - \frac{1}{r}D^\phi + D^\phi_{,r}\right) + \frac{1}{c^2}\left[\frac{\widetilde{W}}{r\sin\theta}D^r_{,\phi}\right.\right. \\
&\quad \left.\left. + (D^\phi_{,r} - \frac{D^\phi}{r})(\widetilde{W} + \frac{1}{2}V^2) + \frac{1}{2}\Omega r\sin\theta D^r_{,T}\right]\right\} + O(4)\,, && \text{(A.7)}\\
e_{\theta\phi} &= r^2\sin\theta\left\{\frac{1}{2}\left(\frac{D^\theta_{,\phi}}{r\sin\theta} - \frac{\cot\theta}{r}D^\phi + \frac{1}{r}D^\phi_{,\theta}\right) + \frac{1}{c^2}\left[\frac{\widetilde{W}}{r\sin\theta}D^\theta_{,\phi}\right.\right. \\
&\quad \left.\left. + \left(\widetilde{W} + \frac{1}{2}V^2\right)\left(\frac{D^\phi_{,\theta}}{r} - \frac{\cot\theta}{r}D^\phi\right) + \frac{1}{2}\Omega r\sin\theta D^\theta_{,T}\right]\right\} + O(4) && \text{(A.8)}
\end{aligned}
$$

where

$$
\mathbf{D}\cdot\nabla\widetilde{W} = \widetilde{W}_{,r}D^r + \frac{1}{r}\widetilde{W}_{,\theta}D^\theta + \frac{1}{r\sin\theta}\widetilde{W}_{,\phi}D^\phi\,.
$$

The volume dilatation Θ, which is related with the expansion rate θ by $\theta = \Theta_{,\mu}U^\mu$, reads

$$
\begin{aligned}
\Theta &= e^\mu{}_\mu = \widetilde{G}^{\mu\nu}e_{\mu\nu} = \widetilde{G}^{ab}e_{ab} + O(4) \\
&= \nabla\cdot\mathbf{D} + \frac{1}{c^2}\left(3\delta\widetilde{W} + 3\mathbf{D}\cdot\nabla\widetilde{W} + \Omega r\sin\theta D^\phi_{,T} + \frac{1}{2}V^2_{,c}\xi^c\right) + O(4)\,, && \text{(A.9)}
\end{aligned}
$$

where

$$
\begin{aligned}
\nabla\cdot\mathbf{D} &\equiv D^r_{,r} + \frac{1}{r}D^\theta_{,\theta} + \frac{1}{r\sin\theta}D^\phi_{,\phi} + \frac{2}{r}D^r + \frac{1}{r}\cot\theta D^\theta\,, \\
\frac{1}{2}V^2_{,c}\xi^c &= \Omega^2 r\sin^2\theta\xi^r + \Omega^2 r^2\sin\theta\cos\theta\xi^\theta = \Omega^2 r\sin^2\theta(D^r + \cot\theta D^\theta)\,.
\end{aligned}
$$

From this it is easy to derive the symmetric trace-free shear tensor $s_{\mu\nu}$. One finds

$$
\begin{aligned}
s_{00} &= O(4)\,, && \text{(A.10)}\\
s_{0r} &= s_{0\theta} = s_{0\phi} = O(5)\,, && \text{(A.11)}\\
s_{rr} &= D^r_{,r} - \frac{1}{3}\nabla\cdot\mathbf{D} + \frac{1}{c^2}\left\{2\widetilde{W}\left(D^r_{,r} - \frac{1}{3}\Theta\right)\right.\\
&\quad \left. -\frac{1}{3}\left(\Omega r\sin\theta D^\phi_{,T} + \Omega^2 r\sin^2\theta(D^r + \cot\theta D^\theta)\right)\right\} + O(4)\,, && \text{(A.12)}\\
s_{\theta\theta} &= r^2\left\{\frac{1}{r}D^\theta_{,\theta} + \frac{1}{r}D^r - \frac{1}{3}\nabla\cdot\mathbf{D} + \frac{1}{c^2}\left[2\widetilde{W}\left(\frac{1}{r}D^\theta_{,\theta} + \frac{1}{r}D^r - \frac{1}{3}\Theta\right)\right.\right.\\
&\quad \left.\left. -\frac{1}{3}\left(\Omega r\sin\theta D^\phi_{,T} + \Omega^2 r\sin^2\theta(D^r + \cot\theta D^\theta)\right)\right]\right\} + O(4)\,, && \text{(A.13)}\\
s_{\phi\phi} &= r^2\sin^2\theta\left\{\frac{1}{r\sin\theta}D^\phi_{,\phi} + \frac{1}{r}D^r + \frac{1}{r}D^\theta\cot\theta - \frac{1}{3}\nabla\cdot\mathbf{D}\right.\\
&\quad +\frac{1}{c^2}\left[(2\widetilde{W} + V^2)\left(\frac{1}{r\sin\theta}D^\phi_{,\phi} + \frac{1}{r}D^r + \frac{1}{r}D^\theta\cot\theta - \frac{1}{3}\Theta\right)\right.\\
&\quad +\Omega^2 r\sin^2\theta\left(D^r + \cot\theta D^\theta\right) + \frac{2}{3}\Omega r\sin\theta D^\phi_{,T}\\
&\quad \left.\left. -\frac{1}{3}\Omega^2 r\sin^2\theta(D^r + \cot\theta D^\theta)\right]\right\} + O(4)\,, && \text{(A.14)}
\end{aligned}
$$

and

$$
s_{r\theta} = e_{r\theta}\,, \qquad s_{r\phi} = e_{r\phi}\,, \qquad s_{\theta\phi} = e_{\theta\phi}\,. \tag{A.15}
$$

These are the post-Newtonian components of the Lagrangian strain and shear tensor in spherical coordinates. If all post-Newtonian terms are neglected (i.e., for $c \to \infty$), they reduce to the well known Newtonian results that can be found in the standard literature.

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